Exact Solutions of Two Body Dirac Equations

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A set of exact eigen values and eigen functions for the two body Dirac problem described by the Hamiltonian

$$H = (\vec{\alpha}_1 - \vec{\alpha}_2)\vec{p} + \beta_1 m_1 + \beta_2 m_2 + \tfrac{1}{2}(\beta_1 + \beta_2)\lambda r$$

are obtained using the properties of Supersymmetric Quantum Mechanics.

Recently the two body Dirac equation has been studied by several authors [1]–[4] as it is the natural way to study a two fermion system and has immediate application in particle physics, particularly in the study of Regge trajectory and light meson spectra.

Very recently, Semay and Ceuleneer [4], [5] have studied a two body Hamiltonian given in the centre of mass frame by (units are such that $\hbar = c = 1$)

$$H = (\vec{\alpha}_1 - \vec{\alpha}_2)\vec{p} + \beta_1 m_1 + \beta_2 m_2 + \frac{1}{2}(\beta_1 + \beta_2)\lambda r, \tag{1}$$

where

$$\vec{p} = -i\vec{\nabla} \text{ and } \vec{r} = \vec{r}_1 - \vec{r}_2. \tag{2}$$

The eigen states of (1) will be 16 component spinors [2], [4]. For central diagonal potentials they can be reduced to simple forms so that one needs only to solve a second order eigenvalue problem involving the radial function $\varphi(r)$ only.

In this note we shall derive a set of exact eigenvalues and eigen functions for the radial wave function problem when λ, the coupling constant, satisfies certain constraint relation. For $l = 0$, we shall verify our results by solving the eigen value problem exactly for all values of λ. Our method is based on the properties of supersymmetric quantum mechanics [6]–[9] which has been used before to calculate exact eigen values [10]–[13] of Schrödinger problems.

The spinor eigen states can be written as [5]

$$\begin{aligned} r\Psi/N_0 = {} & \frac{E + M + \lambda r}{2E}\varphi\,|1; J\,1\,J J_z\rangle \\ & - \frac{1}{E - M'}\varphi'\,|2; J\,1\,J J_z\rangle \\ & + \frac{\sqrt{J(J+1)}}{E - M'}\frac{1}{r}\varphi\,|2; J\,0\,J J_z\rangle \\ & - \frac{1}{E - M'}\varphi'\,|3; J\,1\,J J_z\rangle \\ & + \frac{\sqrt{J(J+1)}}{E + M}\frac{1}{r}\varphi\,|3; J\,0\,J J_z\rangle \\ & - \frac{E - M - \lambda r}{2E}\varphi\,|4; J\,1\,J J_z\rangle, \end{aligned} \tag{3}$$

where N_0 is a normalization factor, $|i; l\,s\,J J_z\rangle$ are angular basis states and M and M' are defined as $M = m_1 + m_2$ and $M' = m_1 - m_2$. The natural parity ($l = J$) radial equation for $\varphi(r)$ is

$$\varphi'' + \left[\frac{(E^2 - (M + \lambda r)^2)(E^2 - M'^2)}{4E^2} - \frac{l(l+1)}{r^2}\right]\varphi = 0. \tag{4}$$

Before casting (4) in super symmetric form we give below a summary of the salient features of super symmetric quantum mechanics (SUSYQM) in one dimension. In one dimension the Hamiltonian of SUSYQM is given by

$$H^{\mathrm{S}} = \{Q^+, Q\} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \tag{5}$$

where

$$H_\pm = -\frac{1}{2}\mathrm{d}^2/\mathrm{d}x^2 + V_\pm(x), \tag{6}$$

$$V_\pm = \frac{1}{2}\left(W^2(x) \pm \mathrm{d}W(x)/\mathrm{d}x\right). \tag{7}$$

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$W(x)$ is called the superpotential and Q, Q^+ the supercharges whose explicit forms are

$$Q = \frac{1}{\sqrt{2}}(p - iW)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{8}$$

$$Q^+ = \frac{1}{\sqrt{2}}(p + iW)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{9}$$

The relations obeyed by Q, Q^+, and H^S are the following:

$$[H^S, Q] = [H^S, Q^+] = 0,$$
$$Q^2 = Q^{+2} = 0.$$

The eigen states of H^S are

$$\varphi^n(x) = \begin{pmatrix} \varphi_+^n(x) \\ \varphi_-^n(x) \end{pmatrix}. \tag{10}$$

If supersymmetry is unbroken, the ground-state energy is zero and the ground-state wave functions are of the form

$$\varphi^n(x) = \begin{pmatrix} \varphi_+^0(x) \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ \varphi_-^0(x) \end{pmatrix} \tag{11}$$

depending on the normalisability of $\varphi_+^0(x)$ or $\varphi_-^0(x)$. Now if $|\Psi\rangle$ is the ground state then

$$Q|\Psi\rangle = Q^+|\Psi\rangle = 0. \tag{12}$$

From (8) and (9) it follows that

$$\varphi_\pm^0(x) = \exp\left(\pm\int^x W(t)\mathrm{d}t\right). \tag{13}$$

For (4) a suitable ansatz for W is

$$W = ar + \frac{b}{r} + d + \sum_{i=1}^{n} \frac{g_i}{1 + g_i r}. \tag{14}$$

We now write (4) as

$$\varphi'' - (V^+(r) - E^+)\varphi = 0, \tag{15}$$

where

$$V^+(r) = W^2 + W'$$

and E^+ is the corresponding eigenvalue for W^2+W' (bosonic sector say). Comparing (15) and (4) and using (14) we get, (equating $W^2 + W' - E^+$ with the term within brackets in (4)),

$$b = l + 1 \tag{16a}$$

$$a = -\frac{\lambda}{2}\left(1 - \frac{M'^2}{E^2}\right)^{3/2}, \tag{16b}$$

$$d = \frac{M\lambda}{4a}\left(1 - \frac{M'^2}{E^2}\right) = -\sum_{i=1}^{n} g_i, \tag{16c}$$

$$2ab + a + 2an + d^2 - E^+ = \frac{1}{4}\left(M^2 + M'^2 - E^2 - \frac{M^2 M'^2}{E^2}\right), \tag{16d}$$

and

$$2dg_i - 2bg_i^2 - 2a + 2\sum_{j\neq i}^{n} \frac{g_i^2 g_j}{g_i - g_j} = 0, \quad i = 1, 2, 3, \cdots. \tag{16e}$$

When $n \leq 2$ the above equations can be solved analytically. We consider the following cases (we assume that m_1 and m_2 are not simultaneously zero).

(i) $n = 1$.

It can be easily seen from (16e) that

$$g_1^2 = -\frac{a}{b+1} \tag{17}$$

The other parameters can be found by solving (16a) – (16c). Eliminating a, d, and b from (16d) and taking $E^+ = 0$ (it corresponds to the ground state of the SUSY potential) we get

$$E^2 = M^2(2l + 5)(l + 2), \tag{18}$$

where we have neglected the solution $E^2 = M'^2$. Further, λ satisfies the relation

$$\lambda = M^2\frac{l+2}{2}\left(1 - \frac{M'^2}{M^2(2l+5)(l+2)}\right)^{1/2}. \tag{19}$$

The wave function is given by

$$\varphi(r) = r^{l+1}(1 + g_1 r)e^{-ar^2/2+dr}, \tag{20}$$

where

$$d = -g_1 = \frac{a}{l+2} \tag{20a}$$

and

$$a = -\frac{\lambda}{2}\left(1 - \frac{M'^2}{E^2}\right)^{3/2}. \tag{20b}$$

(ii) $n = 2$.

Here we first solve g_1, g_2 using (16e). In fact one solves for $g_1 + g_2$ and $g_1 g_2$ and gets

$$g_1 + g_2 = \sqrt{-\frac{a\,(4l+9)}{(l+2)(l+3)}} \tag{21a}$$

and

$$g_1 g_2 = -\frac{a}{l+3}. \tag{21b}$$

Then from (16b) and (16c) one gets

$$\lambda = \frac{M^2}{2}\sqrt{1-\frac{M'^2}{E^2}\,\frac{(l+2)(l+3)}{4l+9}} \tag{21c}$$

and

$$a = -\frac{M^2}{4}\left(1-\frac{M'^2}{E^2}\right)\frac{(l+2)(l+3)}{4l+9}, \tag{21d}$$

and E is given by

$$E^2 = M^2\left[3+\frac{(2l+7)(l+2)(l+3)}{4l+9}\right]. \tag{22}$$

To check our results for $l = 0$ we solve (4) putting

$$x = \frac{(M+\lambda r)^2}{\beta}, \tag{23}$$

where β is given by

$$\beta = \frac{2E\lambda}{\sqrt{E^2 - M'^2}}. \tag{24}$$

Equation (4) reduces to (for $l = J = 0$)

$$x\Psi'' + \left(\frac{1}{2} - x\right)\Psi' - \bar{a}\Psi = 0, \tag{25}$$

where

$$\Psi(x) = e^{x/2}\varphi(x) \tag{26}$$

and

$$\bar{a} = -\frac{E(E^2 - M'^2)^{1/2}}{8\lambda} + \frac{1}{4}. \tag{27}$$

The general solution of (25) is given [14] by

$$\varphi = AM\left(\bar{a}, \frac{1}{2}, x\right) + BM\left(\bar{a}+\frac{1}{2}, \frac{3}{2}, x\right), \tag{28}$$

where A, B are normalization factors and $M(a, b, x)$ is the confluent hypergeometric function. $M\left(\bar{a}, \frac{1}{2}, x\right)$ and $M\left(\bar{a}+\frac{1}{2}, \frac{3}{2}, x\right)$ are respectively the even and odd parity solutions (the parity refers to the variable x, not r). For $N = 1$ we take the even parity solution. The eigenvalues are given by the roots of $M\left(\bar{a}, \frac{1}{2}, x_0\right)$ where $x_0 = x(r = 0)$ i.e. $M\left(\bar{a}, \frac{1}{2}, x_0\right) = 0$, when

$$x_0 = M^2/\beta. \tag{29}$$

To check our exact results with the analytic solutions (26) (for $l = 0$) let us assume that λ takes a value as to make $\bar{a} = -1$. Then $M\left(\bar{a}, \frac{1}{2}, x_0\right) = 0$ gives

$$1 - 2x_0 = 0$$

or

$$x_0 = \frac{M^2}{\beta} = \frac{1}{2}. \tag{30}$$

From the definition of $\bar{a}$ and x_0 we get two simultaneous equations for λ and E, viz.

$$E(E^2 - M'^2)^{1/2} = 10\lambda \tag{31}$$

and

$$M^2 = \frac{E\lambda}{\sqrt{E^2 - M'^2}}. \tag{32}$$

Solving for E and λ we get

$$E^2 = 10M^2$$

and

$$\lambda = M^2\left(1 - \frac{M'^2}{10M^2}\right)^{1/2},$$

which are the same as those given by (18) and (19), respectively, provided we take $l = 0$.

For $n = 0$, we take the odd parity solution, viz. $M\left(\bar{a}+\frac{1}{2}, \frac{3}{2}, x\right)$. If one takes $\bar{a}+\frac{1}{2} = -1$ and $M\left(\bar{a}+\frac{1}{2}, \frac{3}{2}, x_0\right) = 0$ (which gives $x_0 = \frac{3}{2}$) one again gets two simultaeous equations for E and λ. Solving them one gets E and λ given by (22) and (21c), respectively, provided one puts $l = 0$ in these equations.

For the particular case $m_1 = m_2 = 0$ (which means $M = M' = 0$) the potential becomes a function of r^2 only, and hence n in (14) can only have even values. The super potential then should have the terms

$$\sum_{i=1}^{n}\frac{g_i}{1+g_i r} \text{ replaced by } \sum_{i=1}^{n}\frac{2g_i r}{1+g_i r^2}.$$

For example, if one takes $n = 2v$ and $M = M' = 0$, $E^{+} = 0$ in (16d) one gets, using (16b) and (16c)

$$E = \sqrt{2\lambda(4v + 2l + 3)},$$

which is identical with the result obtained in [4] and [5].

To conclude, we have used the properties of SUSYQM to obtain two sets of exact eigenvalues and eigen functions for the Hamiltonian given by (1) when λ satisfies a constraint relation. These solutions would act as bench marks against which the accuracy of numerical and analytical solutions can be judged.

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